

## Exercise 2: Cliques and Parties

*[external exercise not from A&W team, solution on website]*

At a party, every pair of guests is connected by either a **blue** edge (the two guests know each other) or a **red** edge (they do not). A *monochromatic  $k$ -clique* is a set of  $k$  guests such that every pair among them shares the same color edge. For good party dynamics, the host wants to avoid any monochromatic clique: no group of  $k$  guests should all know each other, and no group of  $k$  guests should all be strangers.

**Part (a)** The host wants to invite  $n = 10$  guests. Is it possible to arrange the connections (i.e. to 2-color the edges of  $K_{10}$ ) such that there is *no* monochromatic 5-clique?

Prove your answer, or exhibit an explicit coloring that witnesses it.

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**Part (a)** The host wants to invite  $n = 10$  guests. Is it possible to arrange the connections (i.e. to 2-color the edges of  $K_{10}$ ) such that there is *no* monochromatic 5-clique?

Prove your answer, or exhibit an explicit coloring that witnesses it.

**Part (b)** Generalize the setting of Part (a). Given integers  $n \geq 1$  and  $k \geq 2$ , we say that a party of  $n$  guests is  *$k$ -clique-free* if there exists a 2-coloring of the edges of  $K_n$  with no monochromatic  $k$ -clique.

## Exercise 2: Cliques and Parties - Solution

**Part (a):** Yes, such a colouring exists. We argue via the **probabilistic method**. Colour each edge of  $K_{10}$  independently and uniformly at random. For a fixed set  $S$  of 5 guests, the probability that  $S$  forms a monochromatic 5-clique is

$$\Pr[S \text{ is monochromatic}] = 2 \cdot \left(\frac{1}{2}\right)^{\binom{5}{2}} = 2^{1-10} = \frac{1}{512}.$$

By the union bound over all  $\binom{10}{5} = 252$  such sets,

$$\Pr[\exists \text{ monochromatic 5-clique}] \leq \binom{10}{5} \cdot \frac{1}{512} = \frac{252}{512} = \frac{63}{128} < 1.$$

Since this probability is strictly less than 1, there **must exist** at least one colouring of  $K_{10}$  with no monochromatic 5-clique.  $\square$

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**Part (b):** We generalise the argument to arbitrary  $n$  and  $k$ . Colour each edge of  $K_n$  independently and uniformly at random. For a fixed  $k$ -element subset  $S$ , all  $\binom{k}{2}$  edges must share the same colour, so

$$\Pr[S \text{ is monochromatic}] = 2^{1-\binom{k}{2}}.$$

Applying the union bound over all  $\binom{n}{k}$  subsets of size  $k$ ,

$$\Pr[\exists \text{ monochromatic } k\text{-clique}] \leq \binom{n}{k} \cdot 2^{1-\binom{k}{2}}.$$

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If this quantity is strictly less than 1, a  $k$ -clique-free colouring exists, hence  $R(k, k) > n$ . Using  $\binom{n}{k} \leq \frac{n^k}{k!}$  and  $\binom{k}{2} = \frac{k(k-1)}{2}$ , the condition holds whenever

$$\frac{n^k}{k!} \cdot 2^{1 - \frac{k(k-1)}{2}} < 1.$$

For  $n = 2^{k/2}$  the left-hand side tends to 0 as  $k \rightarrow \infty$ , giving

$$R(k, k) > 2^{k/2}$$

for all sufficiently large  $k$ : a  $k$ -clique-free party of up to  $2^{k/2}$  guests is always achievable.  $\square$

## Exercise 2: Cliques and Parties - Solution

You can find an animated explanation for this question at:  
<https://youtu.be/4weMmFZSBtI>